Then, h is called the ***step-length*** (or interval of differencing).

Therefore $x_1 = x_0 + h$, $x_2 = x_1 + h\, x_0 + 2h, \dots$;

i.e., $$x_r = x_0 + rh, \qquad r = 1, 2, 3, \dots n.$$

Given the values $y_0, y_1, y_2, \dots y_n$ of $y = f(x)$ at $x_0, x_1, x_2, \dots x_n$, let it is required to estimate the value of y corresponding to a desired value x_p of x that lies between x_0 and x_n. One of the ways of making this estimation is to assume that $f(x)$ may be approximated by a polynomial $g(x)$ whose values coincide with the values of $f(x)$ at x_0, x_1, x_2, x_n. It can be proved that this polynomial $g(x)$ is given as follows:

$$g(x) = y_0 + \frac{(x-x_0)}{h}(\Delta y_0) + \frac{(x-x_0)(x-x_1)}{2h^2}(\Delta^2 y_0) + \frac{(x-x_0)(x-x_1)(x-x_2)}{3!\,h^3}(\Delta^3 y_0) + \dots =$$

$$+ \frac{(x-x_0)(x-x_1)\dots(x-x_{n-1})}{n!\,h^n}(\Delta^n y_0) \qquad \dots\dots(i)$$

Taking $x \equiv x_p = x_0 + ph$ in this expression and with

$$x_p - x_r = (x_0 + ph) - (x_0 + rh) = ph - rh = (p - r)h$$

we get the following formula which gives an estimated (approximate) value y_p of $y = f(x)$ at the point $x_p = x_0 + ph$:

$$y_p = y_0 + p\,(\Delta y_0) + \frac{p(p-1)}{2!}(\Delta^2 y_0) + \frac{p(p-1)(p-2)}{3!}(\Delta^3 y_0) +$$

$$\text{-----} + \frac{p(p-1)(p-2)\dots(p-n+1)}{n!}(\Delta^n y_0) \qquad \dots\dots(ii)$$

This formula is known as the **Newton's** or **Newton-Gregory forward interpolation (difference) formula.** This formula is used when $x_p = x_0 + h$ is near the beginning value x_0 of x in the Table for the function $y = f(x)$.

The formula (*ii*) is obtained by approximately $f(x)$ by the polynomial $g(x)$, given by (*i*). Due to this the interpolation carried out by the use of the formula (*ii*) is referrcd to as *polynomial interpolation*. Also, the polynomial $g(x)$ is called the ***interpolating polynomial.***

SOLVED PROBLEMS

1. ***Given***

x	0.1	0.2	0.3	0.4
$f(x)$	2.68	30.4	3.38	3.68

find f(0.15).

Solution. The given values of x are equally space with step-length $h = 0.1$. Using these given values of x we have to determine $f(0.15)$, *i.e.* we have to determine

$$y_p = f(x_p), \text{ where } x_p = 0.15.$$

At first from the following difference table :

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0.1	2.68			
		0.36		
0.2	3.04		– 0.02	
		0.34		– 0.02
0.3	3.38		– 0.04	
		0.30		
0.4	3.68			

From the above Table, with $x_0 = 0.1$, we get

$y_0 = f(0.1) = 2.68, \quad \Delta y_0 = \Delta f(0.1) = 0.36, \Delta^2 y_0 = \Delta^2 f(0.1) = -0.02, \quad \Delta^3 y_0 = \Delta^3 f(0.1) = -0.02.$

Now, taking $x_p = x_0 + ph$, we get $\quad p = \dfrac{x_p - x_0}{h} = \dfrac{0.15 - 0.1}{0.1} = 0.5$

From the forward difference formula (*ii*), we get

$$f(0.15) = y_p = 2.68 + (0.5)(0.36) + \frac{(0.5)(0.5-1)}{2!}(-0.02) + \frac{(0.5)(0.5-1)(0.5-2)}{3!}(-0.02)$$

$$= 2.68 + 0.18 + 0.0025 - 0.00125 = 2.86125 \qquad \text{(approx)}$$

The following table gives the distance in miles of the visible horizon for the given heights in feet above the earth's surface.

x :	*200*	*250*	*300*	*350*	*400*
f(x) :	*15.04*	*16.81*	*18.42*	*19.9*	*21.27*

Find y for x = 218.

Solution. Step-length, $h = 50$. Now find $y_p = f(x_p)$, where $x_p = 218$.

At first form the folloiwng difference Table:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
200	15.04				
		1.77			
			– 0.16	0.03	
250	16.81			↓	
		1:61			– 0.01
			– 0.13		
300	18.42			0.02	
		1.48			
			– 0.11		
350	19.9				
		1.37			
400	21.27				

From the Table, with $x_0 = 200$, we get

$$y_0 = 15.04,\ \Delta y_0 = 17.7,\quad \Delta^2 y_0 = -0.16,\quad \Delta^3 y_0 = 0.03,\quad \Delta^4 y_0 = 0.01.$$

Now, taking $x_p = x_0 + ph$, we get

$$p = \frac{x_p - x_0}{h} = \frac{218 - 200}{50} = 0.36$$

Now, the forward difference formula gives

$$y_p = 15.04 + (0.36)(1.77) + \frac{(0.36)(0.36-1)}{2!}(-16) + \frac{(0.36)(0.36-1)(0.36-2)}{3!}(0.03)$$

$$+ \frac{(0.36)(0.36-1)(0.36-2)(0.36-3)}{4!}(0.01)\ 15.04 + 0.6372 + 0.0184 + 0.0019 + 0.0004$$

= 15.6979

Thus, $y = 15.7$ (approx.) for $x = 218$.

3. ***From the data given in the following Table, find the number of students who obtained***
(i) less than or equal to 45 marks, and
(ii) between 41 and 45 marks.

Marks	***0-40***	***41-50***	***51-60***	***61-70***	***71-80***
Number of students	***31***	***42***	***51***	***35***	***31***

Solution. Let $y = f(x)$ denote the number of students getting $\leq x$.

Then, $y = 31$ for $x = 40$,

$y = 31 + 42 = 73$ for $x = 50$,

$y = 73 + 51 = 124$ for $x = 60$,

$y = 124 + 35 = 159$ for $x = 70$

and $y = 159 + 31 = 190$ for $x = 80$.

Thus the given data may be expressed as shown in the following Table:

x	40	50	60	70	80
y	31	73	124	159	190

The difference Table for this data, where $h = 10$, is as shown below.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
40	31				
		42			
50	73		9		
		51		−25	
60	124		−16		37
		35		12	
70	159		−4		
		31			
80	190				

From the Table, with $x_0 = 40$, we get

$$y_0 = 31, \quad \Delta y_0 = 42, \quad \Delta^2 y_0 = 9, \quad \Delta^3 y_0 = 25, \quad \Delta^4 y_0 = 37.$$

For $x_p = x_0 + ph = 45$, we have $p = \frac{x_p - x_0}{h} = \frac{45 - 40}{10} = 0.5.$

The forward interpolation now gives

$$y_p = f(45) = 31 + (0.5) \times 42 + \frac{(0.5)(0.5-1)}{2} \times 9 + \frac{(0.5)(0.5-1)(0.05-2)}{3!} \times (-25)$$

$$+ \frac{(0.5)(0.5-1)(0.5-2)(0.5-3)}{4!} \times 37$$

$$= 31 + 21 - 11.125 - 1.5625 - 1.445 = 47.8676$$

$$\approx 48.$$

Hence number of stuedents who obtained less than or euqal to 45 marks is about 48. Hence the number of students who obtained less than or equal to 40 marks is 31 (by data), it follows that the number of students who obtained between 41 and 45 marks

$$= 48 - 31 = 17 \text{ (approx.)}$$

4. ***From the following table of values of $y = f(x)$, find the values of y for $x = 1$ and $x = 2$.***

x	3	4	5	6	7	8	9
y	4.8	8.4	14.5	23.6	36.2	52.8	73.9

Solution. The difference table for the given data, for which $h = 1$, is as given below:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
3	4.8				
		3.6			
4	8.4		2.5		
		6.1		0.5	
5	14.5		3.0		0.0
		9.1		0.5	
6	23.6		3.5		0.0
		12.6		0.5	
7	36.2		4.0		0.0
		16.6		0.5	
8	52.8		4.5		
		21.1			
9	73.9				

From the table, with $x_0 = 3$, we get

$$y_0 = 4.8, \quad \Delta y_0 = 3.6, \quad \Delta^2 y_0 = 2.5, \quad \Delta^3 y_0 = 0.5, \quad \Delta^4 y_0 = 0.$$

For $x = x_0 + ph$, we have $p = \frac{x - x_0}{h} = \frac{x - 3}{1} = x - 3$

The forward difference formula gives the corresponding value of $y = f(x)$ as

$$y = f(x) = 4.8 + (x-3) \times (3.6) + \frac{(x-3)(x-4)}{2!} \times (2.5) + \frac{(x-3)(x-4)(x-5)}{3!} \times (0.5)$$

$$= 4.8 + (3.6) \times (x-3) + (1.25) \times (x^2 - 7x + 12 + (0.0833) \times (x^3 - 12x^2 + 47x - 60)$$

$$= (0.0833)\, x^3 + (0.2504)\, x^2 - (1.2349)\, x + 4.002 \quad \text{.........}(i)$$

The right hand side of the above expression is the interpolating polynomial for the given $f(x)$.
For $x = 1$ and $x = 2$, equation (i) gives $y = 3.1008$ and $y = 3.2002$ respectively.

5. ***The table below gives the distance in nautical miles of the visible horizon for the given heights in feet above the earth's surface.***

x = height :	*100*	*150*	*200*	*250*	*300*	*350*	*400*
y = distance:	*10.63*	*13.03*	*15.04*	*16.81*	*18.42*	*19.90*	*21.27*

(Kerala, 1985)

Find the values of y when x = 218 ft

Solution. The difference table is as shown below :

x	y	Δ	Δ^2	Δ^3	Δ^4
100	10.63				
		2.40			
150	13.03		– 0.39		
		2.01		0.15	
200	15.04		– 0.24		– 0.07
		1.77		0.08	
250	16.81		– 0.16		– 0.05
		1.61		0.03	
300	18.42		– 0.13		– 0.01
		1.48		0.02	
350	19.90		– 0.11		
		1.37			
400	21.27				

If we take $x_0 = 200$, then $y_0 = 15.04$, $\Delta y_0 = 1.77$, $\Delta^2 y_0 = -0.16$, $\Delta^3 y_0 = 0.03$ etc.

Since $x = 218$ and $h = 50$, Therefore $n = \frac{18}{50} = 0.36$.

$\therefore$ Using Newton's forward interplation formula, we get

$$f(218) = 15.04 + 0.36(1.77) + \frac{0.36(-0.64)}{2}(-0.16) + \frac{0.36(-0.64)(-1.64)}{6}(0.03) + \ldots$$

$$= 15.04 + 0.637 + .018 + 0.001 + \ldots$$

$$= 15.696 \text{ i.e. } 15.7 \text{ nautical miles.}$$

3.9 NEWTON-GREGORY BACKWARD INTERPOLATION FORMULA

The forward difference formula given in privious section is not suitable for interpolation at a point near the ending value x_n of x in the Table for the function $y = f(x)$.

That the interpolating polynomial that determines an approximate value of $f(x)$ near the ending value x_n of x is given as follows :

$$g(x) = y_n + (x - x_n)\,\nabla y_n + (x - x_n)(x - x_{n-1})\,\frac{\nabla^2 y_n}{h^2(2\,!)} + \ldots$$

$$+ (x - x_n)(x - x_{n-1}) \ldots (x - x_1)\,\frac{\nabla^n y_n}{h^n\,(n\,!)} \qquad \ldots\ldots(i)$$

For $x \equiv x_p = x_n + ph$, the value of $g(x)$ calculated from the above formula give an estimated (approximate) value y_p of $y = f(x)$ at the point $x_p = x_n + ph$.

$$\text{Thus,} \quad y = y_n + p(\nabla y_n) + \frac{p(p+1)}{2!}(\nabla^2 y_n) + \frac{p(p+1)(p+2)}{3!}(\nabla^3 y_n) + \ldots$$

$$\ldots + \frac{p(p+1)(p+2)\ldots(p+n-1)}{n\,!}(\nabla^n y_n) \qquad \ldots\ldots(ii)$$

This formula is known as the **Newton's** or **Newton-Gregory backward interpolation (difference) formula.** This formula is used for interpolation at a point $x_p = x_n + ph$ which is near the ending value x_n of x in the Table for the function $y = f(x)$.

SOLVED PROBLEMS

1. ***The areas y of circles for different diameters x are given below:***

x	80	85	90	95	100
y	5026	5674	6362	7088	7854

Calculate the area when x = 98.

Solution. For the given data, the step length is $h = 5$, and the difference table is as shown below:

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
80	5026				
		648			
85	5674		40		
		688		−2	
90	6362		38		4
		726		2	
95	7088		40		
		766			
100	7554				

From the Table, with $x_n = 100$, we get

$$y_n = y(100) = 7854, \quad \nabla y_n = 766, \quad \nabla^2 y_n = 40, \quad \nabla^3 y_n = 2, \quad \nabla^4 y_n = 4.$$

Let y_p be the value of y at the desired value $x_p = 98$ of x.

Taking $x_p = x_n + ph$, we have $p = \frac{x_p - x_n}{h} = \frac{98-100}{5} = -0.4.$

Now the backward interpolation formula (*ii*) gives

$$y_p = 7854 + (-0.4) \times 766 + \frac{(-0.4+1)}{2!} \times 40 + \frac{(-0.4)(-0.4+1)(-0.4+2)}{3!} \times 2 + \frac{(-0.4)(-0.4+1)(-0.4+2)(-0.4+3)}{4!} \times 4$$

$$= 7854 - 306.4 - 4.8 - 0.128 - 0.1664 = \mathbf{7542.5056}$$

This is the estimated area when $x = 98$.

2. ***Using Newton's backwaed interpolation formula, find the interpolating polynomial that approximates the function given by the following table:***

x :	*0*	*1*	*2*	*3*
f(x) :	*1*	*3*	*7*	*13*

Hence find f(2.5).

Solution. For the given data, the step length, $h = 1$ and the difference Table is as shown below :

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$
0	1			
		2		
1	3		2	
		4		0
2	7		2	
		6		
3	13			

From the Table, with $x_n = 3$, that we get

$$y_n = f(x_n) = 13, \quad \nabla y_n = 6, \quad \nabla^2 y_n = 2, \quad \nabla^3 y_n = 0.$$

For $x = x_n + ph$, we have $p = \frac{x - x_n}{h} = \frac{x-3}{1} = (x-3)$

and the backward interpolation formula gives the corresponding value of y as given below :

$$y = 13 + 6(x-3) + \frac{2}{2!}(x-3)(x-3+1) = x^2 + x + 1.$$

Thus, $g(x) = x^2 + x + 1$ is the interpolating polynomial that approximates to the given function.

For $x = 2.5$, we get $\quad g(2.5) = (2.5)^2 + 2.5 + 1 = 9.75.$

Hence, $\quad f(2.5) \approx g(2.5) = \mathbf{9.75.}$

19.9 LAGRANGE'S INTERPOLATION FORMULA

Let $y_0, y_1, y_2, \ldots y_n$ be the values of an unknown function $y = f(x)$ corresponding to the values $x_0, x_1, x_2, \ldots x_n$ of x, where $x_1, x_2, \ldots x_n$ are *not necessarily* equally spaced. Let $f(x)$ may be approximated by a polynomial of degree n whose values coincide with the values of $f(x)$ at the poins $x_0, x_1, x_2, \ldots x_n$. The polynomial representation of y is given by

$$y = f(x) \approx \frac{(x-x_1)(x-x_2)(x-x_3)\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)\ldots(x_0-x_n)} y_0$$

$$+ \frac{(x-x_0)(x-x_2)(x-x_3)\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)\ldots(x_1-x_n)} y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)\ldots(x-x_n)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)\ldots(x_2-x_n)} y_2$$

$$+ \ldots + \frac{(x-x_0)(x-x_1)\ldots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\ldots(x_n-x_{n-1})} y_n$$

This formula is know as the **Lagrange's interpolation formula.** The polynomial on the right-hand side of this formula is called *Lagrange's interpolating polynomial.*

SOLVED PROBLEMS

1. ***Using Lagrange's interpolation formula, find f(5) from the following data:***

x	1	3	4	6	9
$f(x)$	– 3	9	30	132	156

Solution. Here, $x_0 = 1, \quad x_1 = 3, \quad x_2 = 4, \quad x_3 = 6, \quad x_4 = 9$

$y_0 = -3, \quad y_1 = 9, \quad y_2 = 30, \quad y_3 = 132, \quad y_4 = 156$

Using these values and taking $x = 5$ in the Lagrange's interpolation fromula, we get

$$f(5) = \frac{(5-3)(5-4)(5-6(5-9)}{(1-3)(1-4)(1-6)(1-9)} \times (-3) + \frac{(5-1)(5-4)(5-6)(5-9)}{(3-1)(3-4)(3-6)(3-9)} \times 9$$

$$+ \frac{(5-1)(5-3)(5-6)(5-9)}{(4-1)(4-3)(4-6)(4-9)} \times 30 + \frac{(5-1)(5-3)(5-4)(5-9)}{(6-1)(6-3)(6-6)(6-9)} \times 132$$

$$+ \frac{(5-1)(5-3)(5-4)(5-6)}{(9-1)(9-3)(9-4)(9-6)} \times 156$$

$= -0.1 - 4 + 32 + 46.93 - 1.73 = \mathbf{73.1.}$

2. ***Using the Lagrange's formula, find the interpolation polynomial that approximates to the function described the following table:***

x	0	1	2	3	4
$f(x)$	3	6	11	18	27

Here find f(0.5) and f(3.1).

Solution. Here, $x_0 = 0,\quad x_1 = 1,\quad x_2 = 2,\quad x_3 = 3\quad x_4 = 4$

$y_0 = 3,\quad y_1 = 6,\quad y_2 = 11,\quad y_3 = 18,\quad y_4 = 27$

Putting these values in the Lagrange's interpolation formula, for any x, we get

$$f(x) = \frac{(x-1)(x-2)(x-3)(x-4)}{(0-1)(0-2)(0-3)(0-4)}(3) + \frac{(x-0)(x-2)(x-3)(x-4)}{(1-0)(1-2)(1-3)(1-4)}(6)$$

$$+\frac{(x-0)(x-1)(x-3)(x-4)}{(2-0)(2-1)(2-3)(2-4)}(11) + \frac{(x-0)(x-1)(x-2)(x-4)}{(3-0)(3-1)(3-2)(3-4)}(18)$$

$$+\frac{(x-0)(x-1)(x-2)(x-3)}{(4-0)(4-1)(4-2)(4-3)}(27)$$

$$= \frac{1}{8}(x-1)(x-2)(x-3)(x-4) - x(x-2)(x-3)(x-4)$$

$$+\frac{11}{4}x(x-1)(x-3)(x-4) - 3x(x-1)(x-2)(x-4) + \frac{27}{24}x(x-1)(x-2)(x-3)$$

or
$$\frac{f(x)}{x(x-1)(x-2)(x-3)(x-4)} = \frac{1}{8x} - \frac{1}{x-1} + \frac{11}{4(x-2)} - \frac{3}{x-3} + \frac{9}{8(x-4)}$$

$$= \frac{x^2+2x+3}{x(x-1)(x-2)(x-4)}$$

$\therefore \qquad f(x) = x^2 + 2x + 3$

This is the required polynomial.

$\therefore \qquad f(0.5) = 4.25 \quad \text{and} \quad f(3.1) = 18.81.$

3. *Given the values*

x	5	7	11	13	17
$f(x)$	150	392	1452	2366	5202

Evaluate f(9), using Lagrange's and Newton's divided difference formulae.

Solution.

(*i*) Here $x_0 = 5,\quad x_1 = 7,\quad x_2 = 11,\quad x_3 = 13,\quad x_4 = 17$

and $y_0 = 150,\quad y_1 = 392,\quad y_2 = 1452,\quad y_3 = 2366,\quad y_4 = 5202.$

Putting $x = 9$ and the above values in Lagrange's formula, we get

$$f(9) = \frac{(9-7)(9-11)(9-13)(9-17)}{(5-7)(5-11)(5-13)(5-17)} \times 150 + \frac{(9-5)(9-11)(9-13)(9-17)}{(7-5)(7-11)(7-13)(7-17)} \times 392$$

$$+\frac{(9-5)(9-7)(9-13)(9-17)}{(11-5)(11-7)(11-13)(11-17)} \times 1452 + \frac{(9-5)(9-7)(9-11)(9-17)}{(13-5)(13-7)(13-11)(13-17)} \times 2366$$

$$+\frac{(9-5)(9-7)(9-11)(9-13)}{(17-5)(17-7)(17-11)(17-13)} \times 5202$$

$$= -\frac{50}{3} + \frac{3136}{15} + \frac{3872}{3} - \frac{2366}{3} + \frac{578}{5} = 810.$$

(*ii*) The divided differences table is

x	y	*1st divided differences*	*2nd divided differences*	*3rd divided differences*
5	150			
		121		
7	392		24	
		265		1
11	1452		32	
		457		1
13	2366		42	
		709		
17	5202			

Taking $x = 9$ in the Newton's divided difference formula, we get

$$f(9) = 150 + (9-5) \times 121 + (9-5)(9-7) \times 24 + (9-5)(9-7)(9-11) \times 1$$
$$= 150 + 484 + 192 - 16 = 810.$$

4. ***Apply Bessel's formula to find a polynomial that approximates to the following data:***

x	4	6	8	10
$y(x)$	1	3	8	20

Hence find y(7).

Solution. Here, $h = 2$. Now us take $x_0 = 6$, therefore $y_0 = 3$, and construct the following difference table.

$y(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$
$y_{-1} = 1$			
	2		
$y_0 = 3$		3	
	5		4
$y_1 = 8$		7	
	12		
$y_2 = 20$			

From the table, we get

$$\Delta y_0 = 5, \quad \Delta^2 y_{-1} = 3, \quad \Delta^2 y_0 = 7, \quad \Delta^3 y_{-1} = 4.$$

Putting these values with $p = \frac{x - x_0}{h} = \frac{x-6}{2}$ in the Bessel's formula, we get

$$y(x) = 3 + \left(\frac{x-6}{2}\right) \times 5 + \frac{\left(\frac{x-6}{2}\right)\left(\frac{x-6}{2} - 1\right)}{4}(3+7) + \frac{\left(\frac{x-6}{2}\right)\left(\frac{x-6}{2} - \frac{1}{2}\right)\left(\frac{x-6}{2} - 1\right)}{6} 4$$

$$= 3 + \frac{5}{2}(x-6) + \frac{5}{8}(x-6)(x-8) + \frac{1}{12}(x-6)(x-7)(x-8)$$

$$= \frac{1}{24}[2x^3 - 27x^2 + 142x - 240]$$

This is the required polynomial that approximates to the given data.

$$y(7) = \frac{1}{24}[(2 \times 7^3) + (142 \times 7) - 240] = 4.875.$$

9.12 Inverse Interpolation

Consider the process of finding a value of x that corresponds to a desired value of y. This process is known as *inverse interpolation.*

The Lagrange's interpolation formula given in previous section can be employed for inverse interpolation also, because, this formula is just a relation between the corresponding values of two variables x and y, either of which can be treated as an independent variable. Thus, if we interchange x and y in this formula, we get

$$x = \frac{(y-y_1)(y-y_2)\ldots(y-y_n)}{(y_0-y_1)(y_0-y_2)\ldots(y_0-y_n)} x_0$$

$$+ \frac{(y-y_0)(y-y_2)\ldots(y-y_n)}{(y_1-y_0)(y_1-y_2)\ldots(y_1-y_n)} x_1 + \ldots + \frac{(y-y_0)(y-y_1)\ldots(y-y_{n-1})}{(y_n-y_0)(y_n-y_1)\ldots(y_n-y_{n-1})} x_n$$

This formula gives the value x corresponding to a desired value of y. This is known as **Lagrange's inverse interpolation formula.**

SOLVED PROBLEMS

1. *Given:*

x :	2	5	9	11
y :	10	12	15	19

find x corresponding to $y = 16$.

Solution. Given : $x_0 = 2,\ x_1 = 5,\ x_2 = 9, x_4 = 11,$
$y_0 = 10,\ y_1 = 12, y_2 = 15,\ y_4 = 19$

Hence, the Lagrange's inverse interpolation formula with $y = 16$ gives

$$x = \frac{(16-12)(16-15)(16-19)}{(10-12)(10-15)(10-19)} \times 2 + \frac{(16-10)(16-15)(16-19)}{(12-10)(12-15)(12-19)} \times 5$$

$$+ \frac{(16-10)(16-12)(16-19)}{(15-10)(15-12)(15-19)} \times 9 + \frac{(16-10)(16-12)(16-15)}{(19-10)(19-12)(19-15)} \times 11$$

$$= \frac{24}{90} + \frac{90}{42} + \frac{48}{60} + \frac{264}{252} = \mathbf{9.97143}$$

Thus for the given data, $x = 9.97143$ correspond to $y = 16$.

2. ***Apply Lagrange's method to find the value of x correspopnding to f(x) = 15 from the following data:***

x :	*5*	*6*	*9*	*11*
f(x) :	*12*	*13*	*14*	*16*

Solution. Given : $x_0 = 5$, $x_1 = 6$, $x_2 = 9$, $x_3 = 11$

$y_0 = 2$, $y_1 = 13$, $y_2 = 14$, $y_3 = 16$

Hence, the Lagrange's inverse interpolation formula with $y = f(x) = 15$ gives

$$x = \frac{(15-13)(15-14)(15-16)}{(12-13)(12-14)(12-16)} \times 5 + \frac{(15-12)(15-14)(15-16)}{(13-12)(13-14)(13-16)} \times 16$$

$$+ \frac{(15-12)(15-13)(15-16)}{(14-12)(14-13)(14-16)} \times 9 + \frac{(15-12)(15-13)(15-14)}{(16-12)(16-13)(16-14)} \times 11$$

$$= \frac{54}{4} + \frac{66}{24} = \mathbf{11.5}$$

Thus, $f(x) = 15$ when $x = 11.5$.

3. ***Given:***

x :	*2*	*2.2*	*2.4*	*2.6*	*2.8*
f(x) :	*− 0.6*	*− 0.45*	*− 0.29*	*− 0.12*	*0.05*

find x for which f(x) = 0.

Solution. Given: $x_0 = 2$, $x_1 = 2.2$, $x_2 = 2.4$, $x_3 = 2.6$, $x_4 = 2.8$

$y_0 = -0.6$, $y_1 = -0.45$, $y_2 = -0.29$, $y_3 = -0.12$, $y_4 = 0.05$

Now, Lagrange's formula for $y = f(x) = 0$,

$$x = \frac{0.45)(0.29)(0.12)(-0.05) \times 2}{(-0.6+0.45)(-0.6+0.29)(-0.6+0.12)(-0.6-0.05)}$$

$$+ \frac{(0.6)(0.29)(0.12)((-0.05) \times 2.2}{(-0.45+0.6)(-0.45+0.29)(-0.45+0.12)(-0.45-0.05)}$$

$$+ \frac{(0.6)(0.45)(0.12)(-0.05) \times 2.4}{(-0.29+0.6)(-0.29+0.45)(-0.29+0.12)(-0.29-0.05)}$$

$$+ \frac{(0.6)(0.45)(0.29)(-0.05) \times 2.6}{(-0.12+0.6)(-0.12+0.45)(-0.12+0.29)(-0.12-0.05)}$$

$$+ \frac{(0.6)(0.45)(0.29)(0.12) \times 2.8}{(0.05+0.6)(0.05+0.45)(0.05+0.29)(0.05+0.12)}$$

$$= -0.1079 + 0.58 - 1.3562 + 2.2236 + 1.4005 \approx 2.74.$$

Thus, $f(x) = 0$ when $x \approx \mathbf{2.74}$.

4. ***Apply Lagrange's formula to find a root of the equation $f(x) = 0$, given that $f(30) = -30$, $f(34) = -13$, $f(38) = 3$, $f(42) = 18$.***

Solution. Given:

$$x_0 = -30, \quad y_0 = f(30) = -30,$$
$$x_1 = 34, \quad y_1 = f(34) = -13,$$
$$x_2 = 38, \quad y_2 = f(38) = 3,$$
$$x_4 = 42, \quad y_4 = f(42) = 18,$$

It is required x corresponding to $y\, f(x) = 0$. The Lagrange's formula for $y = 0$, gives

$$x = \frac{13 \times (-3) \times (-18)}{(-30+13) \times (-30-3) \times (-30-18)} \times 30 + \frac{30 \times (-3) \times (-18)}{(-13+30) \times (-13-3) \times (-13-18)} \times 34$$

$$+ \frac{30 \times (-3) \times (-18)}{(-13+30) \times (-13-3) \times (-13-18)} \times 34 + \frac{30 \times 13 \times (-18)}{(3+30) \times (3+13) \times (3-18)} \times 38$$

$$+ \frac{30 \times 13 \times (-3)}{(18+30) \times (18+3) \times (18-3)} \times 42$$

$= 0.78 + 6.53 + 33.68 - 2.2 = \mathbf{37.23}$.

This is the (approximate) root of the equation $f(x) = 0$.

19.13 NUMERICAL DIFFERENTIATION

Consider the application of the Newton's forward and backward interpolation formulas to evaluate the derivaties of functions. The process of finding *approximate values* of the derivatives by using interpolation formulas is called *Numerical Differentiation.*

Derivatives using the Forward Interpolation formula

The Newton's interpolation formula is given by

$$y = y_0 + p\,(\Delta y_0) + \frac{p(p-1)}{2!}(\Delta^2 y_0) + \frac{p(p-1)(p-2)}{3!}(\Delta^3 y_0) + \frac{p(p-1)(p-2)(p-3)}{4!}(\Delta^4 y_0) + \ldots \quad \ldots(i)$$

which gives an approximate value of $y = f(x)$ at $x = x_0 + ph$.

Differentiating equation (*i*) with respect to p term by term, we get

$$y' = (\Delta y) + \frac{2p-1}{2!}(\Delta^2 y_0) + \frac{3p^2-6p+2}{3!}(\Delta^3 y_0) + \frac{4p^3-18p^2+22p-6}{4!}(\Delta^4 y_0) + \ldots \quad \ldots(ii)$$

$$\therefore \quad y' = \frac{dy}{dx} = \frac{dy}{dp}\frac{dp}{dx} = \frac{dy}{dp}\frac{1}{h} \qquad \left(\text{because } \frac{dp}{dx} = \frac{1}{h}\right)$$

$$= \frac{1}{h}\left[(\Delta y_0) + \frac{2p-1}{2}(\Delta^2 y_0) + \frac{3p^2-6p+2}{6}(\Delta y_0) + \frac{4p^3-18p^2+22p-6}{24}(\Delta^4 y_0)\ldots\right] \quad \ldots(iii)$$

This gives y' at any point $x = x_0 + ph$.

At the point x_0, we have $p = 0$, and equation (*iii*) becomes

$$\left(\frac{dy}{dx}\right)(x_0) = \frac{1}{h}\left[(\Delta y_0) - \frac{1}{2}(\Delta^2 y_0) + \frac{1}{3}(\Delta^3 y_0) - \frac{1}{4}(\Delta^4 y_0) + \ldots\right] \qquad \ldots(iv)$$

This formula may be used to compute y' at any of the values of x *where y is specified.*

Again, $$y'' = \frac{d^2y}{dx^2} = \frac{d}{dp}\left(\frac{dy}{dx}\right)\frac{dp}{dx} = \frac{1}{h}\frac{d}{dp}\left(\frac{dy}{dx}\right)$$

From equation (iii)

$$y'' = \frac{1}{h^2}\left[(\Delta^2 y_0) + (p-1)(\Delta^3 y_0) + \frac{6p^2 - 18p + 11}{12}(\Delta^4 y_0) + \ldots\right] \qquad \ldots(v)$$

This gives y'' at any point $x = x_0 = ph$.
At the point x_0 (where $p = 0$), equation (v) becomes

$$\left(\frac{d^2y}{dx^2}\right)(x_0) = \frac{1}{h^2}\left[(\Delta^2 y_0) - (\Delta^3 y_0) + \frac{11}{12}(\Delta^4 y_0) + \ldots\right] \qquad \ldots(vi)$$

This formula may be used to comptue y'' at any of the values of x *where y is specified.*
Similarly, by differentiating equation (v) successively we get formulas for y''', $y^{(iv)}$, ...

SOLVED PROBLEMS

1. ***Given:***

x :	1.0	1.2	1.4	1.6	1.8	2.0
y :	2.72	3.32	4.06	4.96	6.05	7.39

find y' and y'' at $x = 1.2$.

Solution. Here, the step-length $h = 0.2$. At first form the following Difference Table.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
1.0	2.72					
		0.60				
1.2	3.32		0.14			
		0.74		0.02		
1.4	4.06		0.16		0.01	
		0.90		0.03		0.02
1.6	4.96		0.19		0.03	
		1.09		0.06		
1.8	6.05		0.25			
		1.34				
2.0	7.39					

To compute y' and y'' at $x = 1.2$, which is a *specified value* of x, we take, $x_0 = 1.2$. Then we find from the Table

$$\Delta y_0 = 0.74, \quad \Delta^2 y_0 = 0.16, \quad \Delta^3 y_0 = 0.03, \quad \Delta^4 y_0 = 0.03.$$

Then, formula (iv) gives

$$\left(\frac{dy}{dx}\right)(1.2) \approx \frac{1}{(0.2)}\left[(0.74) - \frac{1}{2}(0.16) + \frac{1}{3}(0.03) - \frac{1}{4}(0.03)\right] = 3.3125$$

and formula (*vi*) gives

$$\left(\frac{d^2y}{dx^2}\right)(1.2) \approx \frac{1}{(0.2)^2}\left[(0.16) - (0.03) + \frac{11}{12}(0.03)\right] = 3.9375$$

2. ***A function $y = f(x)$ is specified by the following Table :***

x :	*1*	*1.2*	*1.4*	*1.6*	*1.8*	*2.00*
y :	*0.00*	*0.128*	*0.544*	*1.296*	*2.432*	*4.00*

Find the approximate values of $f'(1.1)$ and $f''(1.1)$.

Solution. Here, the step length, $h = 0.2$ and it is required to find y' and y'' at $x = 1.1$ which is not a tabulated (specified) value of x. The formulas (*iii*) and (*v*) are used to find these derivatives.

At first form the following Difference Table.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1	0.00				
		0.128			
1.2	0.128		0.288		
		0.416		0.048	
1.4	0.544		0.336		0.00
		0.752		0.048	
1.6	1.296		0.384		0.00
		1.136		0.948	
1.8	2.432		0.432		
		1.568			
2.0	4.00				

With $x_0 = 1$, from the Table, we get

$$\Delta y_0 = 0.128, \quad \Delta^2 y_0 = 0.288, \quad \Delta^3 y_0 = 0.048, \quad \Delta^4 y_0 = 0.$$

For $x = x_0 + ph = 1.1$, we have $p = \frac{1.1 - x_0}{h} = \frac{1.1 - 1.0}{0.2} = 0.5.$

Now, from formula noting that $\Delta^4 y_0 = 0$ (*iii*), we get

$$\frac{dy}{dx}(1.1) = \frac{1}{(0.2)}\left[(0.128) + \frac{(2 \times 0.5 - 1)}{2}(0.288) + \frac{3 \times (0.5)^2 - 6 \times (0.5) + 2}{6}(0.048)\right]$$

$$= \frac{1}{(0.2)}[0.128 - 0.02] = 0.63$$

Again, from formula (v), we get

$$\frac{d^2y}{dx^2}(1.1) = \frac{1}{(0.2)^2}[(0.288) + (0.5 - 1) . (0.048)] = \frac{1}{(0.2)^2}[0.288 - 0.24] = 6.6.$$

$f'(1.1) = 0.63$ and $f''(1.1) = $ **6.6.**

3. ***A slider in a machine moves along a fixed straight rod. Its distance x cm along the rod is given below for various values for the time t seconds. Find the velocity of the slider (and its acceleration) when t = 0.3 second.***

t = 0	*0.1*	*0.2*	*0.3*	*0.4*	*0.5*	*0.6*
x = 30.13	*31.62*	*32.87*	*33.64*	*33.95*	*33.81*	*33.24*

(*Gujarat, 1986*)

Solution. As t = 0.2 is the middle value, we will use the central difference formulae noting that a = 0.3 and h = 0.1. Then

$\mu\delta f(0.2) = \frac{1}{2}(0.77 + 0.31) = 0.54$, $\quad \mu\delta^3 f(0.3) = \frac{1}{2}(0.02 + 0.01) = 0.015$,

$$\therefore \quad f'(0.3) = \frac{1}{0.1}\left[0.54 + \frac{1}{6}(0.015) + \frac{1}{30}(-0.125)\right] = 5.34$$

Also $\quad \delta^2 f(0.3) = -0.46, \quad \delta^4 f(0.3) = -0.01, \quad \delta^6 f(0.3) = 0.29.$

$$\therefore \quad f''(0.3) = \frac{1}{(0.1)^2}\left[-0.46 - \frac{1}{12}(-0.01) + \frac{1}{90}(0.29)\right] = \mathbf{-4.56}$$

Hence the required velocity is 5.34 cm/sec and acceleration is – 4.56 cm/sec^2.

9.14 Derivatives using the backward Interpolation formula

From the Newton's backward interpolation formula

$$y = y_n + p(\nabla y_n) + \frac{p(p+1)}{2!}(\nabla^2 y_n) + \frac{p(p+1)(p+2)}{3!}(\nabla^3 y_n) + \frac{p(p+1)(p+2)(p+3)}{4!}(\nabla^4 y_n) + \ldots$$

(i)